The spectra of selfadjoint extensions of entire operators with deficiency indices (1,1)

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Abstract

We give necessary and sufficient conditions for real sequences to be the spectra of selfadjoint extensions of an entire operator whose domain may be non-dense. For this spectral characterization we use de Branges space techniques and a generalization of Krein's functional model for simple, regular, closed, symmetric operators with deficiency indices (1,1). This is an extension of our previous work in which similar results were obtained for densely defined operators.

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1 Introduction

The aim of this work is to present a generalization of the spectral characterization of entire operators given in [18]. This generalization is realized by extending the notion of entire operators to a subclass of symmetric operators with deficiency indices $(1, 1)$ that may have non-dense domain. The spectral characterization of a given operator in the class is based on the distribution of the spectra of its selfadjoint extensions within the Hilbert space. More concretely, for a given simple, regular, closed symmetric (possibly not densely defined) operator with deficiency indices $(1, 1)$ to be entire it is necessary and sufficient that the spectra of two of its selfadjoint extensions satisfy conditions which reduce to the convergence of certain series (the precise statement is Proposition 5.2).

The class of entire operators was concocted by M. G. Krein as a tool for treating in a unified way several classical problems in analysis [10–12, 14]. The entire operators form a subclass of the closed, densely defined, symmetric, regular operators with equal deficiency indices. They have many remarkable properties as is accounted for in the review book [7]. Krein's definition of entire operators hinges on his functional model for symmetric operators and it requires the existence of an element of the Hilbert space with very peculiar properties. As first discussed in [18] it is possible to determine whether an operator is entire by conditions that rely exclusively on the distribution of the spectra of selfadjoint extensions of the operator.

Although Krein's original work considers only densely defined symmetric operators, it is clear that the definition of entire operators can be extended to the case of not necessarily dense domain with no formal changes (see Definition 2.5). Since non-densely and densely defined symmetric operators share certain properties, the machinery developed in [18] carries over with some mild modifications.

One ingredient of our discussion is an extension of the functional model developed in [18]. This functional model associates a de Branges space to every simple, regular, closed symmetric operator with deficiency indices (1,1). It is worth remarking that functional models for this and for related classes of operators have been implemented before; see for instance [5,20]. However, the functional model proposed in [18] has shown to be particularly suitable for us. Here we deem appropriate to mention [16] for a related kind of results.

This paper is organized as follows. In Section 2 we recall some of the properties held by operators that are closed, simple, symmetric with deficiency indices $(1, 1)$; the notion of entire operator is also introduced here. Section 3 provides a short review on the theory of de Branges Hilbert spaces, including those results relevant to this work, in particular, a slightly modified version of a theorem due to Woracek (Proposition 3.1). In Section 4 we introduce a functional model for any operator of the class under consideration so that the model space is always a de Branges space. Finally, in Section 5 we single out the class of

de Branges spaces corresponding to entire operators and provide necessary and sufficient conditions on the spectra of two selfadjoint extensions of an entire operator.

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2 On symmetric operators with not necessarily dense domain

Let $\mathcal{H}$ be a separable Hilbert space whose inner product $\langle \cdot, \cdot \rangle$ is assumed antilinear in its first argument. In this space we consider a closed, symmetric operator A with deficiency indices $(1, 1)$. It is not assumed that its domain is dense in $\mathcal{H}$, therefore one should deal with the case when the adjoint of A is a linear relation. That is, in general,

$$A^* := \{\{\eta, \omega\} \in \mathcal{H} \oplus \mathcal{H} : \langle \eta, A\varphi \rangle = \langle \omega, \varphi \rangle \text{ for all } \varphi \in \operatorname{dom}(A)\} . \tag{2.1}$$

Whenever the orthogonal complement of $\operatorname{dom}(A)$ is trivial, the set $A^*(0) := \{\omega \in \mathcal{H} : \{0, \omega\} \in A^*\}$ is also trivial, i. e. $A^*(0) = \{0\}$, so A^* is an operator; otherwise A^* is a proper closed linear relation.

For $z \in \mathbb{C}$ one has

$$A^* - zI := \{\{\eta, \omega - z\eta\} \in \mathcal{H} \oplus \mathcal{H} : \{\eta, \omega\} \in A^*\} \tag{2.2}$$

so accordingly

$$\ker(A^* - zI) := \{\eta \in \mathcal{H} : \{\eta, 0\} \in A^* - zI\} . \tag{2.3}$$

Since $\ker(A^* - zI) = \mathcal{H} \ominus \operatorname{ran}(A - \overline{z}I)$, our assumption on the deficiency indices implies $\dim \ker(A^* - zI) = 1$ for all $z \in \mathbb{C} \setminus \mathbb{R}$. Also, since

$$A^*(0) = \{\omega \in \mathcal{H} : \langle \omega, \psi \rangle = 0 \text{ for all } \psi \in \operatorname{dom}(A)\} ,$$

it is obvious that $A^*(0) = \operatorname{dom}(A)^\perp$.

The selfadjoint extensions within $\mathcal{H}$ of a closed, non-densely defined symmetric operator A are the selfadjoint linear relations that extend the graph of A. We recall that a linear relation B is selfadjoint if $B = B^*$ (as subsets of $\mathcal{H} \oplus \mathcal{H}$).

The following assertion follows easily from [8, Section 1, Lemma 2.2 and Theorem 2.4].

Proposition 2.1. *Let A be a closed, non-densely defined, symmetric operator in $\mathcal{H}$ with deficiency indices $(1, 1)$. Then:*

(i) The codimension of $\operatorname{dom}(A)$ *equals one.*

(ii) All except one of the selfadjoint extensions of A within $\mathcal{H}$ are operators.

(iii) Let A_γ be one of the selfadjoint extensions of A within $\mathcal{H}$. Then the operator

$$I + (z - w)(A_\gamma - zI)^{-1}, \quad z \in \mathbb{C} \setminus \operatorname{spec}(A_\gamma), \quad w \in \mathbb{C}$$

maps $\ker(A^* - wI)$ *injectively onto* $\ker(A^* - zI)$.

In connection with this proposition we remind the reader that the spectrum of a closed linear relation B is the complement of the set of all $z \in \mathbb{C}$ such that $(B - zI)^{-1}$ is a bounded operator defined on all $\mathcal{H}$. Moreover, $\operatorname{spec}(B) \subset \mathbb{R}$ when B is a selfadjoint linear relation [6].

Given $\psi_{w_0} \in \ker(A^* - w_0 I)$, with $w_0 \in \mathbb{C} \setminus \mathbb{R}$, let us define

$$\psi(z) := \left[I + (z - w_0)(A_\gamma - zI)^{-1} \right] \psi_{w_0}, \tag{2.4}$$

Note that $I + (z - w_0)(A_\gamma - zI)^{-1}$ is the generalized Cayley transform. Obviously, $\psi(w_0) = \psi_{w_0}$. Moreover, a computation involving the resolvent identity yields

$$\psi(z) = \left[I + (z - v)(A_\gamma - zI)^{-1} \right] \psi(v), \tag{2.5}$$

for any pair $z, v \in \mathbb{C} \setminus \mathbb{R}$. This identity will be used later on.

Let us now recall some concepts that will be used to single out a class of closed symmetric operators with deficiency indices $(1, 1)$.

A closed, symmetric operator A is called *simple* if

$$\bigcap_{z \in \mathbb{C} \setminus \mathbb{R}} \operatorname{ran}(A - zI) = \{0\}.$$

Equivalently, A is simple if there exists no non-trivial subspace $\mathcal{L} \subset \mathcal{H}$ that reduces A and whose restriction to $\mathcal{L}$ yields a selfadjoint operator [15, Proposition 1.1].

There is one property specific to simple, closed symmetric operators with deficiency indices $(1, 1)$, that is of interest to us. It concerns their commutativity with involutions. We say that an involution J commutes with a selfadjoint relation B if

$$J(B - zI)^{-1}\varphi = (B - \overline{z}I)^{-1} J\varphi,$$

for every $\varphi \in \mathcal{H}$ and $z \in \mathbb{C} \setminus \mathbb{R}$. If B is moreover an operator this is equivalent to the usual notion of commutativity, that is,

$$J \operatorname{dom}(B) \subset \operatorname{dom}(B), \qquad JB\varphi = BJ\varphi$$

for every $\varphi \in \operatorname{dom}(B)$.

Proposition 2.2. *Let A be a simple, closed symmetric operator with deficiency indices $(1,1)$. Then there exists an involution J that commutes with all its selfadjoint extensions within $\mathcal{H}$.*

Proof. Choose a selfadjoint extension A_γ and consider $\psi(z)$ as defined by (2.4). Recalling (2.5) along with the unitary character of the generalized Cayley transform, and applying the resolvent identity, one can verify that

$$\langle \psi(\overline{z}), \psi(\overline{v}) \rangle = \langle \psi(v), \psi(z) \rangle \tag{2.6}$$

for every pair $z, v \in \mathbb{C} \setminus \mathbb{R}$.

Now define the action of J on the set $\{\psi(z) : z \in \mathbb{C} \setminus \mathbb{R}\}$ by the rule

$$J\psi(z) = \psi(\overline{z}),$$

and on the set $\mathcal{D}$ of finite linear combinations of such elements as

$$J\left(\sum_n c_n \psi(z_n)\right) := \sum_n \overline{c_n}\psi(\overline{z_n}).$$

Then, on one hand, (2.6) implies that J is an involution on $\mathcal{D}$ which can be extended to all $\mathcal{H}$ because of the simplicity of A. On the other hand, since by the resolvent identity

$$(A_\gamma - wI)^{-1}\psi(z) = \frac{\psi(z) - \psi(w)}{z - w},$$

one obtains the identity

$$J(A_\gamma - wI)^{-1}\psi(z) = (A_\gamma - \overline{w}I)^{-1}J\psi(z)$$

which by linearity holds on $\mathcal{D}$ and in turn it extends to all $\mathcal{H}$.

So far we know that J commutes with A_γ. By resorting to the well-known resolvent formula due to Krein (see [8, Theorem 3.2] for a generalized formulation), one immediately obtains the commutativity of J with all the selfadjoint extensions of A within $\mathcal{H}$. □

A closed, symmetric operator is called *regular* if for every $z \in \mathbb{C}$ there exists $d_z > 0$ such that

$$\|(A - zI)\psi\| \geq d_z \|\psi\|, \tag{2.7}$$

for all $\psi \in \operatorname{dom}(A)$. In other words, A is regular if every point of the complex plane is a point of regular type.

Definition 2.3. Let $\mathcal{S}(\mathcal{H})$ be the class of simple, regular, closed symmetric operators in $\mathcal{H}$, whose deficiency indices are $(1,1)$.

In [17, 18] we deal with the subclass of operators in $\mathcal{S}(\mathcal{H})$ that are densely defined. In the present work we extend the results of [18] to the larger class defined above. At this point it is convenient to touch upon some well-known properties shared by the operators in $\mathcal{S}(\mathcal{H})$ that are densely defined, and whose generalizations to the whole class is rather straightforward. The following statement is one of such generalizations which we believe may have been already proven, however, due to the lack of the proper reference, we provide the proof below.

Proposition 2.4. *For $A \in \mathcal{S}(\mathcal{H})$ the following assertions hold true:*

(i) The spectrum of every selfadjoint extension of A within $\mathcal{H}$ consists solely of isolated eigenvalues of multiplicity one.

(ii) Every real number is part of the spectrum of one, and only one, selfadjoint extension of A within $\mathcal{H}$.

(iii) The spectra of the selfadjoint extensions of A within $\mathcal{H}$ are pairwise interlaced.

Proof. Let us proof (i) in a way similar to the one used to prove [7, Propositions 3.1 and 3.2], but taking into account that the operator is not necessarily densely defined.

For $A \in \mathcal{S}(\mathcal{H})$ and any $r \in \mathbb{R}$ consider the constant d_r of (2.7). Thus, the symmetric operator $(A-rI)^{-1}$, defined on the subspace $\operatorname{ran}(A-rI)$, is such that $\|(A-rI)^{-1}\| \le d_r^{-1}$. By [13, Theorem 2] there is a selfadjoint extension B of $(A-rI)^{-1}$ defined on the whole space and such that $\|B\| \le d_r^{-1}$. Now, B^{-1} is a selfadjoint extension of $A - rI$ and $\|B^{-1}f\| \ge d_r \|f\|$ for any $f \in \operatorname{dom}(B^{-1})$, which implies that the interval $(-d_r, d_r) \cap \operatorname{spec}(B^{-1}) = \emptyset$. By shifting B^{-1} one obtains a selfadjoint extension of A with no spectrum in the spectral lacuna $(r - d_r, r + d_r)$. By perturbation theory any selfadjoint extension of A which is an operator has no points of the spectrum in this spectral lacuna other than one eigenvalue of multiplicity one. When $\overline{\operatorname{dom}(A)} \neq \mathcal{H}$, the same is also true for the spectrum of the selfadjoint extension which is not an operator. This follows from a generalization of the Aronzajn-Krein formula (see [8, Equation 3.17]) after noting that the Weyl function is Herglotz and meromorphic for any selfadjoint extension being an operator. Now, for proving (i) consider any closed interval of $\mathbb{R}$, cover it with spectral lacunae and take a finite subcover.

Once (i) has been proven, the assertions (ii) and (iii) follow from [8, Equation 3.17] and the properties of Herglotz meromorphic functions. □

Definition 2.5. An operator $A \in \mathcal{S}(\mathcal{H})$ is called *entire* if there exists $\mu \in \mathcal{H}$ such that

$$\mathcal{H} = \operatorname{ran}(A - zI) \dot{+} \operatorname{span}\{\mu\}$$

for all $z \in \mathbb{C}$. Such μ is called an *entire gauge*.

If $A \in \mathcal{S}(\mathcal{H})$ turns out to be densely defined, then Definition 2.5 reduces to Krein's [12, Section 1]. There are various densely defined operators known to be entire [7, Chapter 3], [12, Section 4]. On the other hand, for what will be explained in the subsequent sections, there are also entire operators with non-dense domain. Let us outline how one may construct an entire operator which is not densely defined. The details of this construction will be expounded in a further paper.

Consider the semi-infinite Jacobi matrix

$$\begin{pmatrix} q_1 & b_1 & 0 & 0 & \cdots \\ b_1 & q_2 & b_2 & 0 & \cdots \\ 0 & b_2 & q_3 & b_3 & \\ 0 & 0 & b_3 & q_4 & \ddots \\ \vdots & \vdots & & \ddots & \ddots \end{pmatrix}, \tag{2.8}$$

where $b_k > 0$ and $q_k \in \mathbb{R}$ for $k \in \mathbb{N}$. Fix an orthonormal basis $\{\delta_k\}_{k\in\mathbb{N}}$ in $\mathcal{H}$. Let B be the operator in $\mathcal{H}$ whose matrix representation with respect to $\{\delta_k\}_{k\in\mathbb{N}}$ is (2.8) (cf. [2, Section 47]). We assume that $B \neq B^*$, equivalently, that B has deficiency indices $(1,1)$ [1, Chapter 4, Section 1.2]. Let B_0 be the restriction of B to the set $\{\phi \in \operatorname{dom}(B) : \langle \phi, \delta_1 \rangle = 0\}$. It follows from (2.1), (2.2) and (2.3) that $\eta \in \ker(B_0^* - zI)$ if and only if it satisfies the equation

$$\langle B\phi, \eta \rangle = \langle \phi, z\eta \rangle \qquad \forall \phi \in \operatorname{dom}(B_0)\,.$$

Thus $\ker(B_0^* - zI)$ is the set of η's in $\mathcal{H}$ that satisfy

$$b_{k-1} \langle \delta_{k-1}, \eta \rangle + q_k \langle \delta_k, \eta \rangle + b_k \langle \delta_{k+1}, \eta \rangle = z \langle \delta_k, \eta \rangle \quad \forall k > 1 \tag{2.9}$$

Hence $\dim \ker(B_0^* - zI) \leq 2$. Now, let

$$\pi(z) := \sum_{k=1}^{\infty} P_{k-1}(z) \delta_k \qquad \theta(z) := \sum_{k=1}^{\infty} Q_{k-1}(z) \delta_k\,,$$

where $P_k(z)$, respectively $Q_k(z)$, is the k-th polynomial of first, respectively second, kind associated to (2.8). By the definition of the polynomials $P_k(z)$ and $Q_k(z)$ [1, Chapter 1, Section 2.1], $\pi(z)$ and $\theta(z)$ are linearly independent solutions of (2.9) for every fixed

$z \in \mathbb{C}$. Moreover, since $B \neq B^*$, $\pi(z)$ and $\theta(z)$ are in $\mathcal{H}$ for all $z \in \mathbb{C}$ [1, Theorems 1.3.1, 1.3.2], [19, Theorem 3]. So one arrives at the conclusion that, for every fixed $z \in \mathbb{C}$,

$$\ker(B_0^* - zI) = \operatorname{span}\{\pi(z), \theta(z)\}\,.$$

Any symmetric non-selfadjoint extension of B_0 has deficiency indices (1,1). Furthermore, if $\kappa(z)$ is a (z-dependent) linear combination of $\pi(z)$ and $\theta(z)$ such that $\langle \kappa(z), \theta(z)\rangle = 0$ for all $z \in \mathbb{C} \setminus \mathbb{R}$, then (by a parametrized version of [19, Theorem 2.4]) there corresponds to an appropriately chosen isometry from $\operatorname{span}\{\kappa(z)\}$ onto $\operatorname{span}\{\kappa(\overline{z})\}$ a non-selfadjoint symmetric extension $\widetilde{B}$ of B_0 such that $\operatorname{dom}(\widetilde{B})$ is not dense and $\ker(\widetilde{B}^* - zI) = \operatorname{span}\{\theta(z)\}$. We claim that $\widetilde{B}$ is a non-densely defined entire operator. Indeed, $\widetilde{B} \in \mathcal{S}(\mathcal{H})$ (the simplicity follows from the properties of the associated polynomials [1, Chapter 1, Addenda and Problems 7]). Moreover, since

$$\langle \theta(z), \delta_2 \rangle = b_1^{-1}\,, \qquad \forall z \in \mathbb{C}\,,$$

δ_2 is an entire gauge.

3 A review on de Branges spaces with zero-free functions

Let $\mathcal{B}$ denote a nontrivial Hilbert space of entire functions with inner product $\langle \cdot, \cdot \rangle_{\mathcal{B}}$. $\mathcal{B}$ is a de Branges space when, for every function $f(z)$ in $\mathcal{B}$, the following conditions holds:

(A1) For every $w \in \mathbb{C} \setminus \mathbb{R}$, the linear functional $f(\cdot) \mapsto f(w)$ is continuous;

(A2) for every non-real zero w of $f(z)$, the function $f(z)(z - \overline{w})(z - w)^{-1}$ belongs to $\mathcal{B}$ and has the same norm as $f(z)$;

(A3) the function $f^{\#}(z) := \overline{f(\overline{z})}$ also belongs to $\mathcal{B}$ and has the same norm as $f(z)$.

It follows from (A1) that for every non-real w there is a function $k(z, w)$ in $\mathcal{B}$ such that $\langle k(\cdot, w), f(\cdot)\rangle_{\mathcal{B}} = f(w)$ for all $f(z) \in \mathcal{B}$. Moreover, $k(w, w) = \langle k(\cdot, w), k(\cdot, w)\rangle_{\mathcal{B}} \geq 0$ where, as a consequence of (A2), the positivity is strict for every non-real w unless $\mathcal{B}$ is $\mathbb{C}$; see the proof of Theorem 23 in [4]. Note that $k(z, w) = \langle k(\cdot, z), k(\cdot, w)\rangle_{\mathcal{B}}$ whenever z and w are both non-real, therefore $k(w, z) = \overline{k(z, w)}$. Furthermore, due to (A3) it can be shown that $\overline{k(\overline{z}, w)} = k(z, \overline{w})$ for every non-real w; we refer again to the proof of Theorem 23 in [4]. Also note that $k(z, w)$ is entire with respect to its first argument and, by (A3), it is anti-entire with respect to the second one (once $k(z, w)$, as a function of its second argument, has been extended to the whole complex plane [4, Problem 52]).

There is another way of defining a de Branges space. One starts by considering an entire function $e(z)$ of the Hermite-Biehler class, that is, an entire function without zeros

in the upper half-plane $\mathbb{C}^+$ that satisfies the inequality $|e(z)| > \left|e^{\#}(z)\right|$ for $z \in \mathbb{C}^+$. Then, the de Branges space $\mathcal{B}(e)$ associated to $e(z)$ is the linear manifold of all entire functions $f(z)$ such that both $f(z)/e(z)$ and $f^{\#}(z)/e(z)$ belong to the Hardy space $H^2(\mathbb{C}^+)$, and equipped with the inner product

$$\langle f(\cdot), g(\cdot)\rangle_{\mathcal{B}(e)} := \int_{-\infty}^{\infty} \frac{\overline{f(x)}g(x)}{|e(x)|^2} dx.$$

It turns out that $\mathcal{B}(e)$ is complete.

Both definitions of de Branges spaces are equivalent, viz., every space $\mathcal{B}(e)$ obeys (A1–A3); conversely, given a space $\mathcal{B}$ there exists an Hermite-Biehler function $e(z)$ such that $\mathcal{B}$ coincides with $\mathcal{B}(e)$ as sets and the respective norms satisfy the equality $\|f(\cdot)\|_{\mathcal{B}} = \|f(\cdot)\|_{\mathcal{B}(e)}$ [4, Chapter 2]. The function $e(z)$ is not unique; a choice for it is

$$e(z) = -i\sqrt{\frac{\pi}{k(w_0, w_0)\operatorname{im}(w_0)}}\,(z - \overline{w_0})\, k(z, w_0),$$

where w_0 is some fixed complex number in $\mathbb{C}^+$.

An entire function $g(z)$ is said to be associated to a de Branges space $\mathcal{B}$ if for every $f(z) \in \mathcal{B}$ and $w \in \mathbb{C}$,

$$\frac{g(z)f(w) - g(w)f(z)}{z - w} \in \mathcal{B}.$$

The set of associated functions is denoted $\operatorname{assoc}\mathcal{B}$. It is well known that

$$\operatorname{assoc}\mathcal{B} = \mathcal{B} + z\mathcal{B};$$

see [4, Theorem 25] and [9, Lemma 4.5] for alternative characterizations. In passing, let us note that $e(z) \in \operatorname{assoc}\mathcal{B}(e) \setminus \mathcal{B}(e)$; this fact follows easily from [4, Theorem 25].

The space $\operatorname{assoc}\mathcal{B}(e)$ contains a distinctive family of entire functions. They are given by

$$s_\beta(z) := \frac{i}{2}\left[e^{i\beta}e(z) - e^{-i\beta}e^{\#}(z)\right], \quad \beta \in [0, \pi).$$

These real entire functions are related to the selfadjoint extensions of the multiplication operator S defined by

$$\operatorname{dom}(S) := \{f(z) \in \mathcal{B} : zf(z) \in \mathcal{B}\}, \quad (Sf)(z) = zf(z). \tag{3.10}$$

This is a simple, regular, closed symmetric operator with deficiency indices $(1, 1)$ which is not necessarily densely defined [9, Proposition 4.2, Corollary 4.3, Corollary 4.7]. It turns out that $\overline{\operatorname{dom}(S)} \neq \mathcal{B}$ if and only if there exists $\gamma \in [0, \pi)$ such that $s_\gamma(z) \in \mathcal{B}$.

Furthermore, $\operatorname{dom}(S)^\perp = \operatorname{span}\{s_\gamma(z)\}$ [4, Theorem 29] and [9, Corollary 6.3]; compare with (i) of Proposition 2.1.

For any selfadjoint extension $S_\sharp$ of S there exists a unique β in $[0,\pi)$ such that

$$(S_\sharp - wI)^{-1} f(z) = \frac{f(z) - \frac{s_\beta(z)}{s_\beta(w)} f(w)}{z-w}, \quad w \in \mathbb{C} \setminus \operatorname{spec}(S_\sharp), \quad f(z) \in \mathcal{B}. \tag{3.11}$$

Moreover, $\operatorname{spec}(S_\sharp) = \{x \in \mathbb{R} : s_\beta(x) = 0\}$. [9, Propositions 4.6 and 6.1]. If $S_\sharp$ is a selfadjoint operator extension of S, then (3.11) is equivalent to

$$\operatorname{dom}(S_\sharp) = \left\{ g(z) = \frac{f(z) - \frac{s_\beta(z)}{s_\beta(z_0)} f(z_0)}{z - z_0}, \quad f(z) \in \mathcal{B}, \quad z_0 : s_\beta(z_0) \neq 0 \right\},$$

$$(S_\sharp g)(z) = z g(z) + \frac{s_\beta(z)}{s_\beta(z_0)} f(z_0).$$

The eigenfunction g_x corresponding to $x \in \operatorname{spec}(S_\sharp)$ is given (up to normalization) by

$$g_x(z) = \frac{s_\beta(z)}{z - x}.$$

Thus, since S is regular and simple, every $s_\beta(z)$ has only real zeros of multiplicity one and the (sets of) zeros of any pair $s_\beta(z)$ and $s_{\beta'}(z)$ are always interlaced.

The proof of the following result can be found in [21] for a particular pair of selfadjoint extensions of S. Another proof, when the operator S is densely defined, is given in [18, Proposition 3.9].

Proposition 3.1. *Suppose $e(x) \neq 0$ for $x \in \mathbb{R}$ and $e(0) = (\sin\gamma)^{-1}$ for some fixed $\gamma \in (0,\pi)$. Let $\{x_n\}_{n\in\mathbb{N}}$ be the sequence of zeros of the function $s_\gamma(z)$. Also, let $\{x_n^+\}_{n\in\mathbb{N}}$ and $\{x_n^-\}_{n\in\mathbb{N}}$ be the sequences of positive, respectively negative, zeros of $s_\gamma(z)$, arranged according to increasing modulus. Then a zero-free, real entire function belongs to $\mathcal{B}(e)$ if and only if the following conditions hold true:*

(C1) The limit $\displaystyle\lim_{r\to\infty} \sum_{0<|x_n|\le r} \frac{1}{x_n}$ *exists;*

(C2) $\displaystyle\lim_{n\to\infty} \frac{n}{x_n^+} = -\lim_{n\to\infty} \frac{n}{x_n^-} < \infty;$

(C3) Assuming that $\{b_n\}_{n\in\mathbb{N}}$ are the zeros of $s_\beta(z)$, define

$$h_\beta(z) := \begin{cases} \lim\limits_{r\to\infty} \prod\limits_{|b_n|\le r} \left(1 - \dfrac{z}{b_n}\right) & \text{if 0 is not a root of } s_\beta(z), \\ z \lim\limits_{r\to\infty} \prod\limits_{0<|b_n|\le r} \left(1 - \dfrac{z}{b_n}\right) & \text{otherwise.} \end{cases}$$

The series $\sum\limits_{n\in\mathbb{N}} \left|\dfrac{1}{h_0(x_n)h'_\gamma(x_n)}\right|$ *is convergent.*

Proof. Combine Theorem 3.2 of [21] with Lemmas 3.3 and 3.4 of [18]. □

4 A functional model for operators in $\mathcal{S}(\mathcal{H})$

The functional model given in this section follows the construction developed in [18], now adapted to include all the operators in the class $\mathcal{S}(\mathcal{H})$. This functional model is based on (the properties of) the operator mentioned in (iii) of Proposition 2.1 with the following addition.

Proposition 4.1. *Given $A \in \mathcal{S}(\mathcal{H})$, let J be an involution that commutes with one of its selfadjoint extensions within $\mathcal{H}$ (hence with all of them), say, A_γ. Choose $v \in \operatorname{spec}(A_\gamma)$. Then, there exists $\psi_v \in \ker(A^* - vI)$ such that $J\psi_v = \psi_v$.*

Proof. Let ϕ_v be an element of $\ker(A_\gamma - vI)$. Since J commutes with A_γ, one immediately obtains that $J\phi_v \in \ker(A_\gamma - vI)$. But, by our assumption on the deficiency indices of A and its regularity, $\ker(A^* - vI)$ is a one dimensional space and it contains $\ker(A_\gamma - vI)$. So, in $\ker(A_\gamma - vI)$, J reduces to multiplication by a scalar α and the properties of the involution imply that $|\alpha| = 1$. Now, $\psi_v := (1 + \alpha)\phi_v$ has the required properties. □

Given $A \in \mathcal{S}(\mathcal{H})$ and an involution J that commutes with its selfadjoint extensions within $\mathcal{H}$, define

$$\xi_{\gamma,v}(z) := h_\gamma(z)\left[I + (z - v)(A_\gamma - zI)^{-1}\right]\psi_v\,, \tag{4.12}$$

where v and ψ_v are chosen as in the previous proposition, and $h_\gamma(z)$ is a real entire function whose zero set is $\operatorname{spec}(A_\gamma)$ (see Proposition 2.4 (i)). Clearly, up to a zero-free real entire function, $\xi_{\gamma,v}(z)$ is completely determined by the choice of the selfadjoint extension A_γ and v. Actually, as it is stated more precisely below, $\xi_{\gamma,v}(z)$ does not depend on A_γ nor on v.

Proposition 4.2. *(i) The vector-valued function $\xi_{\gamma,v}(z)$ is zero-free and entire. It lies in $\ker(A^* - zI)$ for every $z \in \mathbb{C}$.*

(ii) $J\xi_{\gamma,v}(z) = \xi_{\gamma,v}(\overline{z})$ *for all* $z \in \mathbb{C}$.

(iii) *Given* $\xi_{\gamma_1,v_1}(z)$ *and* $\xi_{\gamma_2,v_2}(z)$*, there exists a zero-free real entire function* $g(z)$ *such that* $\xi_{\gamma_2,v_2}(z) = g(z)\xi_{\gamma_1,v_1}(z)$.

Proof. Due to (iii) of Proposition 2.1, the proof of (i) is rather straightforward. In fact, one should only follow the first part of the proof of [18, Lemma 4.1]. The proof of (ii) also follows easily from our choice of ψ_w and $h_\gamma(z)$ in the definition of $\xi_{\gamma,w}(z)$. To prove (iii), one first uses (iii) of Proposition 2.1 and the fact that $\dim\ker(A^* - wI) = 1$ to obtain that $\xi_{\gamma_2,w_2}(z)$ and $\xi_{\gamma_1,w_1}(z)$ differ by a nonzero scalar complex function. Then the reality of this function follows from (ii). $\square$

For the reason already explained, from now on the function $\xi_{\gamma,v}(z)$ will be denoted by $\xi(z)$. Now define

$$(\Phi\varphi)(z) := \langle \xi(\overline{z}), \varphi \rangle\,, \qquad \varphi \in \mathcal{H}.$$

Φ maps $\mathcal{H}$ onto a certain linear manifold $\widehat{\mathcal{H}}$ of entire functions. Since A is simple, it follows that Φ is injective. A generic element of $\widehat{\mathcal{H}}$ will be denoted by $\widehat{\varphi}(z)$, as a reminder of the fact that it is the image under Φ of a unique element $\varphi \in \mathcal{H}$.

The linear space $\widehat{\mathcal{H}}$ is turned into a Hilbert space by defining

$$\langle \widehat{\eta}(\cdot), \widehat{\varphi}(\cdot) \rangle := \langle \eta, \varphi \rangle\ .$$

Clearly, Φ is an isometry from $\mathcal{H}$ onto $\widehat{\mathcal{H}}$.

Proposition 4.3. $\widehat{\mathcal{H}}$ *is a de Branges space.*

Proof. It suffices to show that the axioms given at the beginning of Section 3 holds for $\widehat{\mathcal{H}}$.

It is straightforward to verify that $k(z,w) := \langle \xi(\overline{z}), \xi(\overline{w}) \rangle$ is a reproducing kernel for $\widehat{\mathcal{H}}$. This accounts for (A1).

Suppose $\widehat{\varphi}(z) \in \widehat{\mathcal{H}}$ has a zero at $z = w$. Then its preimage $\varphi \in \mathcal{H}$ lies in $\operatorname{ran}(A - wI)$. This allows one to set $\eta \in \mathcal{H}$ by

$$\eta = (A - \overline{w}I)(A - wI)^{-1}\varphi = \varphi + (w - \overline{w})(A_\gamma - wI)^{-1}\varphi.$$

Now, recalling (4.12) and applying the resolvent identity one obtains

$$\langle \xi(\overline{z}), \eta \rangle = \frac{z - \overline{w}}{z - w} \langle \xi(\overline{z}), \varphi \rangle\,.$$

Since η and φ are related by a Cayley transform, the equality of norms follows. This proves (A2).

As for (A3), consider any $\widehat{\varphi}(z) = \langle \xi(\overline{z}), \varphi \rangle$. Then, as a consequence of (ii) of Proposition 4.2, one has $\widehat{\varphi}^{\#}(z) = \langle \xi(\overline{z}), J\varphi \rangle$. □

It is worth remarking that the last part of the proof given above shows that $^{\#} = \Phi J \Phi^{-1}$.

The following obvious assertion is the key of (every) functional model; we state it for the sake of completeness.

Proposition 4.4. *Let S be the multiplication operator on $\widehat{\mathcal{H}}$ given by (3.10).*

(i) $S = \Phi A \Phi^{-1}$ *and* $\operatorname{dom}(S) = \Phi \operatorname{dom}(A)$.

(ii) The selfadjoint extensions of S within $\widehat{\mathcal{H}}$ are in one-one correspondence with the selfadjoint extensions of A within $\mathcal{H}$.

Item (ii) above can be stated more succinctly by saying that

$$\Phi(A_\beta - zI)^{-1}\Phi^{-1} = (S_\beta - zI)^{-1}, \quad z \in \mathbb{C} \setminus \operatorname{spec}(A_\gamma),$$

for all β of a certain (common) parametrization of the selfadjoint extensions of both A and S. This expression is of course valid even for the exceptional (i.e. non-operator) selfadjoint extension of A. In passing we note that the exceptional selfadjoint extension of a non-densely defined operator in $\mathcal{S}(\mathcal{H})$ corresponds to the selfadjoint extension of the operator S whose associated function lies in $\widehat{\mathcal{H}}$.

5 Spectral characterization

In the previous section we constructed a functional model that associates a de Branges space to every operator A in $\mathcal{S}(\mathcal{H})$ in such a way that the operator of multiplication in the de Branges space is unitarily equivalent to A. The first task in this section is to single out the class of de Branges spaces corresponding to entire operators in our functional model. Having found this class, we use the theory of de Branges spaces to give a spectral characterization of the multiplication operator for the class we found. This is how we give necessary and sufficient conditions on the spectra of two selfadjoint extensions of an entire operator.

The following proposition gives a characterization of the class of de Branges spaces corresponding to entire operators in our functional model.

Proposition 5.1. $A \in \mathcal{S}(\mathcal{H})$ *is entire if and only if* $\widehat{\mathcal{H}}$ *contains a zero-free entire function.*

Proof. Let $g(z) \in \widehat{\mathcal{H}}$ be the function whose existence is assumed. Clearly there exists (a unique) $\mu \in \mathcal{H}$ such that $g(z) \equiv \langle \xi(\overline{z}), \mu \rangle$. Therefore, μ is never orthogonal to $\ker(A^* - zI)$ for all $z \in \mathbb{C}$. That is, μ is an entire gauge for the operator A.

The necessity is established by noting that the image of the entire gauge under Φ is a zero-free function. ☐

Proposition 5.2. *For $A \in \mathcal{S}(\mathcal{H})$, consider the selfadjoint extensions (within $\mathcal{H}$) A_0 and A_γ, with $0 < \gamma < \pi$. Then A is entire with real entire gauge μ ($J\mu = \mu$) if and only if* $\operatorname{spec}(A_0)$ *and* $\operatorname{spec}(A_\gamma)$ *obey conditions (C1), (C2) and (C3) of Proposition 3.1.*

Proof. Apply Proposition 3.1 along with Proposition 5.1. ☐

We remark that when A is an entire operator with non-dense domain, it may be that either A_0 or A_γ is not an operator (see Proposition 2.1 (ii)). Nevertheless, even in this case, $\operatorname{spec}(A_0)$ and $\operatorname{spec}(A_\gamma)$ satisfy (C1), (C2) and (C3).

The following proposition shows, among other things, that the original functional model by Krein is a particular case of our functional model.

Proposition 5.3. *Assume $1 \in \widehat{\mathcal{H}}$. Then there exists $\mu \in \mathcal{H}$ such that*

$$h_\gamma(z) = \left\langle \psi_v + (z - v)(A_\gamma - zI)^{-1}\psi_v, \mu \right\rangle^{-1}$$

and $J\mu = \mu$. Moreover, μ is the unique entire gauge of A modulo a real scalar factor.

Proof. Necessarily, $1 \equiv \langle \xi(\overline{z}), \mu \rangle$ for some $\mu \in \mathcal{H}$. By (4.12), and taking into account the occurrence of J, one obtains the stated expression for $h_\gamma(z)$. By the same token, the reality of μ is shown.

Suppose that there are two real entire gauges μ and μ'. The discussion in Paragraph 5.2 of [7] shows that $(\Phi_\mu \mu')(z) = ae^{ibz}$ with $a \in \mathbb{C}$ and $b \in \mathbb{R}$. Due to the assumed reality, one concludes that $b = 0$ and $a \in \mathbb{R}$. ☐

6 Concluding remarks

We would like to add some few comments concerning further extensions of the present work.

First, since there are de Branges spaces that contain the constant functions but whose multiplication operator is not densely defined, it follows that, apart from the example given in Section 2, there should be other operators in the class introduced in this work that are not comprised in the original Krein's notion of entire operators. The details of our example as well as other ones and applications of our results will be studied elsewhere.

Second, it is possible to define a notion of a (possibly non-densely defined) operator that is entire in a generalized sense, much in the same vein as the original definition by Krein for densely defined operators (see [7, Chapter 2, Section 9]). Following [18, Section

5], operators entire in this generalized sense could also be characterized by the spectra of their selfadjoint extensions.

Finally, it is known that the set of selfadjoint operator extensions within $\mathcal{H}$ of a non-densely defined operator are in one-one correspondence with a set of rank-one perturbations of one of these selfadjoint operator extensions [8, Section 2]. This set of rank-one perturbations is generated by elements in $\mathcal{H}$ so it seems interesting to study the relation (if any) between these elements and the gauges of operators in $\mathcal{S}(\mathcal{H})$. Ultimately, we believe that a suitable characterization of the rank-one perturbations could provide another necessary and sufficient condition for a non-densely defined operator in $\mathcal{S}(\mathcal{H})$ to be entire. This problem, as well as the previous one, will be discussed in a subsequent work.

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